

On the time evolution of a homogeneous isotropic gaseous universe

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November 23, 2015

Abstract

We investigate the time evolution of a homogeneous isotropic universe which consists of a perfect gas governed by the Einstein-Euler-de Sitter equations.. Under suitable assumptions on the equation of state and assumptions on the present states, we give a mathematically rigorous proof of the existence of the Big Bang at the finite past and expanding of the universe in the course to the infinite future.

1 Introduction

We consider the Einstein-Euler-de Sitter equations

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}(g^{\alpha\beta}R_{\alpha\beta})g_{\mu\nu} - \Lambda g_{\mu\nu} &= \frac{8\pi G}{c^4}T_{\mu\nu}, \\ T^{\mu\nu} &= (c^2\rho + P)U^\mu U^\nu - Pg^{\mu\nu} \end{aligned}$$

for the metric

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu.$$

Here the cosmological constant Λ is supposed to be positive. We assume

(A0) P is a smooth function of $\rho > 0$ such that $0 \leq P, 0 \leq dP/d\rho < c^2$ for $\rho > 0$ and $P \rightarrow 0$ as $\rho \rightarrow +0$.

We consider the co-moving spherically symmetric metric

$$ds^2 = e^{2F(t,r)}c^2dt^2 - e^{2H(t,r)}dr^2 - R(t,r)^2(d\theta^2 + \sin^2\theta d\phi^2)$$

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such that $U^0 = e^{-F}$, $U^1 = U^2 = U^3 = 0$ for $x^0 = ct$, $x^1 = r$, $x^2 = \theta$, $x^3 = \phi$. Then the equations on the region where $\rho > 0$ turn out to be

$$e^{-F} \frac{\partial R}{\partial t} = V \quad (1a)$$

$$e^{-F} \frac{\partial \rho}{\partial t} = -(\rho + P/c^2) \left(\frac{V'}{R'} + \frac{2V}{R} \right) \quad (1b)$$

$$e^{-F} \frac{\partial V}{\partial t} = -GR \left(\frac{m}{R^3} + \frac{4\pi P}{c^2} \right) + \frac{c^2 \Lambda}{3} R + \\ - \left(1 + \frac{V^2}{c^2} - \frac{2Gm}{c^2 R} - \frac{\Lambda}{3} R^2 \right) \frac{P'}{R'(\rho + P/c^2)} \quad (1c)$$

$$e^{-F} \frac{\partial m}{\partial t} = -\frac{4\pi}{c^2} R^2 P V. \quad (1d)$$

Here X' stands for $\partial X / \partial r$. We put

$$m = 4\pi \int_0^r \rho R^2 R' dr \quad (2)$$

and the coefficients of the metric are given by

$$P' + (c^2 \rho + P) F' = 0 \quad (3)$$

and

$$e^{2H} = \left(1 + \frac{V^2}{c^2} - \frac{2Gm}{c^2 R} - \frac{\Lambda}{3} R^2 \right)^{-1} (R')^2. \quad (4)$$

For derivation of these equations, see [7], [8].

Now we are going to consider solutions of the form

$$R(t, r) = a(t)r, \quad \rho = \rho(t), \quad F(t, r) = 0.$$

Of course we consider $a(t) > 0$. Then (1a) reads

$$V(t, r) = \frac{da}{dt} r, \quad (5)$$

(2) reads

$$m(t, r) = \frac{4\pi}{3} a(t)^3 r^3 \rho(t), \quad (6)$$

(1c) reads

$$\frac{d^2 a}{dt^2} = \left(-\frac{4\pi G}{3} (\rho + 3P/c^2) + \frac{c^2 \Lambda}{3} \right) a, \quad (7)$$

(1b) reads

$$\frac{d\rho}{dt} = -3(\rho + P/c^2) \frac{1}{a} \frac{da}{dt}, \quad (8)$$

and (1d) is reduced to (8). Therefore we have to consider only (7) and (8).

Let us write (7)(8) as the first order system

$$\frac{da}{dt} = \dot{a}, \quad (9a)$$

$$\frac{d\dot{a}}{dt} = \left(-\frac{4\pi G}{3}(\rho + 3P/c^2) + \frac{c^2\Lambda}{3} \right) a, \quad (9b)$$

$$\frac{d\rho}{dt} = -3(\rho + P/c^2) \frac{\dot{a}}{a}. \quad (9c)$$

We shall investigate the solution $(a(t), \dot{a}(t), \rho(t))$ of the system (9a)(9b)(9c) which satisfies the initial conditions

$$a(0) = a_0, \quad \dot{a}(0) = \dot{a}_0, \quad \rho(0) = \rho_0. \quad (10)$$

Let $]t_-, t_+[$, $-\infty \leq t_- < 0 < t_+ \leq +\infty$, be the maximal interval of existence of the solution in the domain

$$\mathcal{D} = \{(a, \dot{a}, \rho) \mid 0 < a, 0 < \rho\}. \quad (11)$$

We shall use the variable ρ^b , a given function of ρ , defined by

$$\rho^b = \exp \left[\int^\rho \frac{d\rho}{\rho + P/c^2} \right]. \quad (12)$$

The assumption (A0) implies that $\rho \leq \rho + P/c^2 \leq 2\rho$ so that $\rho^b \rightarrow +\infty \Leftrightarrow \rho \rightarrow +\infty$, and $\rho^b \rightarrow 0 \Leftrightarrow \rho \rightarrow 0$. The equation (9c) reads

$$\frac{d\rho^b}{dt} = -3 \frac{\rho^b}{a} \frac{da}{dt},$$

therefore it holds that

$$\frac{\rho^b}{\rho_0^b} = \left(\frac{a_0}{a} \right)^3, \quad (13)$$

where $\rho_0^b = \rho^b|_{\rho=\rho_0}$, as long as the solution exists.

Let us recall the theory of A. Friedman [2] (1922). If (7)(8) hold, then it follows that the quantity

$$X := \left(\frac{da}{dt} \right)^2 - \left(\frac{8\pi G}{3} \rho + \frac{c^2\Lambda}{3} \right) a^2$$

enjoys

$$\frac{dX}{dt} = 0.$$

Therefore there should exist a constant K such that

$$\left(\frac{da}{dt} \right)^2 = \left(\frac{8\pi G}{3} \rho + \frac{c^2\Lambda}{3} \right) a^2 - c^2 K. \quad (14)$$

Of course the constant K is determined by the initial condition as

$$K = \frac{1}{c^2} \left(\left(\frac{8\pi G}{3} \rho_0 + \frac{c^2 \Lambda}{3} \right) (a_0)^2 - (\dot{a}_0)^2 \right).$$

By defining K as this, we can write

$$e^{2H} = (1 - Kr^2)^{-1} a^2$$

and the metric turns out to be

$$ds^2 = c^2 dt^2 - a(t)^2 \left(\frac{dr^2}{1 - Kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right).$$

The equation (14) is nothing but [2, (6)] when $\rho \propto a^{-3}$.

Remark 1. Note that if (8)(14) hold, then we can claim that (7) holds as long as $da/dt \neq 0$. But, when $da/dt = 0$, or, $a(t) = \bar{a} = \text{Const.} > 0$, then $\rho(t)$ should be a constant, say, $\bar{\rho}$ and

$$4\pi G(\bar{\rho} + 3\bar{P}/c^2) = c^2 \Lambda \quad (15)$$

should hold by (7). Then

$$e^{2H} = (1 - k\bar{a}^2 r^2)^{-1} \bar{a}^2$$

with

$$k = \frac{4\pi G}{c^2} (\bar{\rho} + \bar{P}/c^2).$$

In this case, we can put $\bar{a} = 1$ without loss of generality, and the metric turns out to be

$$ds^2 = c^2 dt^2 - \left(\frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right).$$

Since $k > 0$ provided that $\bar{\rho} > 0$, this is the so-called ‘Einstein’s universe (1917)’.

The following rule is trivial:

Proposition 1 *If $(a, \dot{a}, \rho) = (\phi_0(t), \phi_1(t), \psi(t))$ satisfies (9a)(9b) (9c) for $t \in I$, I being an open interval, then $(a, \dot{a}, \rho) = (\phi_0(t + \Theta), \phi_1(t + \Theta), \psi(t + \Theta))$ satisfies (9a)(9b)(9c) for $t \in I - \Theta := \{t | t + \Theta \in I\}$, Θ being a constant, and $(a, \dot{a}, \rho) = (\phi_0(-t), -\phi_1(-t), \psi(-t))$ satisfies (9a)(9b)(9c) for $t \in -I := \{t | -t \in I\}$.*

2 Big Bang

Hereafter we shall consider the solution such that

$$\left. \frac{da}{dt} \right|_{t=0} = \dot{a}_0 > 0, \quad (16)$$

which requires

$$\left(\frac{8\pi G}{3}\rho_0 + \frac{c^2\Lambda}{3}\right)(a_0)^2 > c^2 K. \quad (17)$$

In this section we investigate what will happen when we prolong the solution to the left, or, to the past. We claim

Theorem 1 *Assume (A0). If the initial data satisfy (16) and $\frac{d^2 a}{dt^2}\big|_{t=0} \leq 0$, that is,*

$$4\pi G(\rho_0 + 3P_0/c^2) \geq c^2 \Lambda, \quad (18)$$

then t_- is finite and $a(t) \rightarrow 0, \rho(t) \rightarrow +\infty$ as $t \rightarrow t_- + 0$.

Proof. First we claim that $\dot{a}(t) > 0$ for $t_- < \forall t \leq 0$. Otherwise, there exists $t_1 \in]t_-, 0[$ such that $\dot{a}(t) > 0$ for $t_1 < t \leq 0$ and $\dot{a}(t_1) = 0$, since $\dot{a}(0) = \dot{a}_0 > 0$ is supposed ((16)). Therefore

$$\begin{aligned} \frac{d}{dt}(4\pi G(\rho + 3P/c^2)) &= 4\pi G\left(1 + \frac{3}{c^2} \frac{dP}{d\rho}\right) \frac{d\rho}{dt} = \\ &= -4\pi G\left(1 + \frac{3}{c^2} \frac{dP}{d\rho}\right) \cdot 3(\rho + P/c^2) \frac{\dot{a}}{a} < 0, \end{aligned}$$

for $t_1 < \forall t \leq 0$, and we see that (18) implies

$$4\pi G(\rho + 3P/c^2) \geq c^2 \Lambda$$

for $t_1 < \forall t \leq 0$. Therefore

$$\frac{d}{dt}\left(\frac{8\pi G}{3}\rho + \frac{c^2\Lambda}{3}\right)a^2 = -\frac{2}{3}(4\pi G(\rho + 3P/c^2) - c^2\Lambda)a\dot{a} \leq 0$$

and

$$\left(\frac{8\pi G}{3}\rho(t) + \frac{c^2\Lambda}{3}\right)a(t)^2 \geq \left(\frac{8\pi G}{3}\rho_0 + \frac{c^2\Lambda}{3}\right)a_0^2$$

for $t_1 < \forall t \leq 0$. We are supposing that

$$\delta := (\dot{a}_0)^2 = \left(\frac{8\pi G}{3}\rho_0 + \frac{c^2\Lambda}{3}\right)a_0^2 - c^2 K > 0.$$

Therefore,

$$\left(\frac{8\pi G}{3}\rho(t) + \frac{c^2\Lambda}{3}\right)a(t)^2 - c^2 K \geq \delta$$

for $t_1 < \forall t \leq 0$, which implies $\dot{a}(t)^2 \geq \delta$ for $t_1 < \forall t \leq 0$, a contradiction to $\dot{a}(t_1) = 0$. Hence we see $\dot{a}(t)^2 > 0$ for $t_- < \forall t \leq 0$. Since $\dot{a}(0) > 0$, we have $\dot{a}(t) > 0$ for $t_- < \forall t \leq 0$ and, repeating the above argument, we see $da/dt = \dot{a}(t) \geq \sqrt{\delta}$ for $t_- < \forall t \leq 0$. This implies $t_- > -\infty$. Otherwise, if $t_- = -\infty$, we would have $a(t) \leq a_0 + \sqrt{\delta}t \rightarrow -\infty$ as $t \rightarrow -\infty$, a contradiction. Since $da/dt = \dot{a} > 0, d\rho/dt < 0$, we have $a(t) \rightarrow a_- (\geq 0), \rho(t) \rightarrow \rho_- (\leq +\infty)$ as $t \rightarrow t_- + 0$.

We claim $\rho_- = +\infty$. Otherwise, if $\rho_- < +\infty$, we have $a_- > 0$ thanks to (13). Then the solution could be continued across t_- to the left. Hence it should be that $\rho_- = +\infty$ and $a_- = 0$ thanks to (13). This completes the proof. \square

Since the metric is

$$ds^2 = c^2 dt^2 - a(t)^2 \left(\frac{dr^2}{1 - Kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right),$$

we can consider the volume of the space at $t = \text{Const.}$ is $a(t)^3 \cdot 2\pi^2 / K^{3/2}$, provided that $K > 0$ and the space is the 3-dimensional hypersphere of radius $a(t)/\sqrt{K}$. Since $a(t) \rightarrow 0$ and $\rho(t) \rightarrow +\infty$ as $t \rightarrow t_- + 0$, this is nothing but the so called ‘Big Bang’.

If $K \leq 0$, then the condition (18) can be dropped, that is, we have

Corollary 1 *Suppose (16): $\dot{a}_0 > 0$. If $K \leq 0$, that is, if*

$$\left(\frac{8\pi G}{3} \rho_0 + \frac{c^2 \Lambda}{3} \right) (a_0)^2 \leq (\dot{a}_0)^2, \quad (19)$$

then the conclusion of Theorem 1 holds even if (18) is not satisfied.

In fact, suppose (19). Since we have $\dot{a}(t) > 0$ for $t_- < \forall t \leq 0$ a priori thanks to $K \leq 0$ (see (14)), we have

$$\frac{d\rho}{dt} = -3(\rho + P/c^2) \frac{\dot{a}}{a} < 0$$

for $t_- < \forall t \leq 0$ a priori. Therefore $\rho(t) \rightarrow \rho_- (\leq +\infty)$, $a(t) \rightarrow a_-$ as $t \rightarrow t_-$. Suppose $\rho_- < +\infty$. Note that $\rho_- > \rho_0 > 0$. Then $a_- > 0$ thanks to (13) and $\dot{a}(t)$ tends to a finite limit. Thus, if $t_- > -\infty$, the solution could be continued to the left across t_- , a contradiction. Hence $t_- = -\infty$. Since $\rho_- < +\infty$ is supposed, we should have $a_- > 0$. But (14) and $K \leq 0$ implies

$$\frac{da}{dt} \geq \sqrt{\frac{8\pi G \rho_0}{3}} a \geq \delta := \sqrt{\frac{8\pi G \rho_0}{3}} a_- > 0,$$

which implies $a \leq a_0 + \delta t \rightarrow -\infty$ as $t \rightarrow -\infty$, a contradiction. Therefore it should be the case that $\rho_- = +\infty$. Then there exists $t_0 \in]t_-, 0[$ such that

$$4\pi G(\rho(t_0) + 3P(t_0)/c^2) \geq c^2 \Lambda,$$

and the conclusion of Theorem 1 holds. \square

More precise behavior of $a(t), \rho(t)$ as $t \rightarrow t_- + 0$ can be obtained by an additional assumption on the behavior of the function $\rho \mapsto P$ as $\rho \rightarrow +\infty$. Let us give an example.

The equation of state for neutron stars is given by

$$P = Ac^5 \int_0^\zeta \frac{q^4 dq}{(1+q^2)^{1/2}}, \quad \rho = 3Ac^3 \int_0^\zeta (1+q^2)^{1/2} q^2 dq.$$

See [4, p. 188]. Then we have

$$P = \frac{c^2}{3} \rho (1 + [\rho^{-1/2}]_1) \quad (20)$$

as $\rho \rightarrow +\infty$. Here $[X]_1$ stands for a convergent power series of the form $\sum_{k \geq 1} a_k X^k$.

Thus, generalizing this situation, we assume

(A1) There are constants Γ, σ such that $1 \leq \Gamma < 2, 0 < \sigma \leq 1$ and

$$P = (\Gamma - 1)c^2 \rho (1 + O(\rho^{-\sigma})) \quad (21)$$

as $\rho \rightarrow +\infty$.

Then integrating (9c) gives

$$a = a_1 \rho^{-\frac{1}{3\Gamma}} (1 + O(\rho^{-\sigma}))$$

as $\rho \rightarrow +\infty$ with a positive constant a_1 and

$$\rho = \rho_1 a^{-3\Gamma} (1 + O(a^{3\Gamma\sigma}))$$

as $a \rightarrow 0$ with $\rho_1 = (a_1)^{3\Gamma}$. Then the equation (14) turns out to be

$$\frac{da}{dt} = \sqrt{\frac{8\pi G \rho_1}{3}} a^{\frac{-3\Gamma+2}{2}} (1 + O(a^\nu)),$$

where $\nu := \min(3\Gamma\sigma, 3\Gamma - 2)$. Solving this, we have

$$a = (6\pi\Gamma^2 G \rho_1)^{\frac{1}{3\Gamma}} (t - t_-)^{\frac{2}{3\Gamma}} (1 + O((t - t_-)^{\frac{2\nu}{3\Gamma}})), \quad (22)$$

$$\rho = \frac{1}{6\pi\Gamma^2 G} (t - t_-)^{-2} (1 + O((t - t_-)^{\frac{2\nu}{3\Gamma}})) \quad (23)$$

Remark 2. Suppose that $K > 0$. Then the space at $t = \text{Const.}$ has a finite volume and the total mass is

$$M(t) = \frac{4\pi}{3} a(t)^3 K^{-3/2} \rho(t).$$

However, since (9c) implies

$$\frac{d}{dt} a^3 \rho = -\frac{3P}{c^2} a^2 \frac{da}{dt} \neq 0,$$

provided that $P > 0$, the total mass is not conserved during the time evolution of our universe. Therefore when $P \neq 0$, instead, we should consider the variable ρ^b . Then the modified total mass

$$M^b(t) = \frac{4\pi}{3}a(t)^3 K^{-3/2} \rho^b(t)$$

is constant with respect to t .

Remark 3. Note that the arguments of this section are still valid even if $\Lambda = 0$. If $\Lambda = 0$, (18) is satisfied for any $\rho_0 > 0$, and we have Big Bang whenever (14): $\dot{a}_0 > 0$.

3 Expanding universe

Let us investigate the behavior of the solution when continued to the right (to the future) as long as possible. We claim

Theorem 2 *Assume (A0)(A1). If the initial data satisfy (16) and $\frac{d^2 a}{dt^2}\big|_{t=0} \geq 0$, that is,*

$$4\pi G(\rho_0 + 3P_0/c^2) \leq c^2 \Lambda, \quad (24)$$

then $t_+ = +\infty$ and $a(t) \rightarrow +\infty, \rho(t) \rightarrow 0$ as $t \rightarrow +\infty$.

Proof. First we claim that $\dot{a}(t) > 0$ for $0 \leq \forall t < t_+$. Otherwise, there exists $t_1 \in]0, t_+[$ such that $\dot{a}(t) > 0$ for $0 \leq \forall t < t_1$ and $\dot{a}(t_1) = 0$, since $\dot{a}(0) = \dot{a}_0 > 0$ is supposed ((16)). Therefore

$$\begin{aligned} \frac{d}{dt}(4\pi G(\rho + 3P/c^2)) &= 4\pi G\left(1 + \frac{3}{c^2} \frac{dP}{d\rho}\right) \frac{d\rho}{dt} = \\ &= -4\pi G\left(1 + \frac{3}{c^2} \frac{dP}{d\rho}\right) \cdot 3(\rho + P/c^2) \frac{\dot{a}}{a} < 0, \end{aligned}$$

for $0 \leq \forall t < t_1$, and we see that (24) implies

$$4\pi G(\rho + 3P/c^2) \leq c^2 \Lambda$$

for $0 \leq \forall t < t_1$. Therefore

$$\frac{d}{dt}\left(\frac{8\pi G}{3}\rho + \frac{c^2 \Lambda}{3}\right)a^2 = -\frac{2}{3}(4\pi G(\rho + 3P/c^2) - c^2 \Lambda)a\dot{a} \geq 0$$

and

$$\left(\frac{8\pi G}{3}\rho(t) + \frac{c^2 \Lambda}{3}\right)a(t)^2 \geq \left(\frac{8\pi G}{3}\rho_0 + \frac{c^2 \Lambda}{3}\right)a_0^2$$

for $0 \leq \forall t < t_1$. We are supposing that

$$\delta := (\dot{a}_0)^2 = \left(\frac{8\pi G}{3}\rho_0 + \frac{c^2 \Lambda}{3}\right)a_0^2 - c^2 K > 0.$$

Therefore,

$$\left(\frac{8\pi G}{3}\rho(t) + \frac{c^2\Lambda}{3}\right)a(t)^2 - c^2K \geq \delta$$

for $0 \leq \forall t < t_1$, which implies $\dot{a}(t)^2 \geq \delta$ for $0 \leq \forall t < t_1$, a contradiction to $\dot{a}(t_1) = 0$. Hence we see $\dot{a}(t)^2 > 0$ for $0 \leq \forall t < t_+$. Since $\dot{a}(0) > 0$, we have $\dot{a}(t) > 0$ for $0 \leq \forall t < t_+$ and, repeating the above argument, we see $da/dt = \dot{a}(t) \geq \sqrt{\delta}$ for $0 \leq \forall t < t_+$.

We claim $t_+ = +\infty$. In fact, suppose $t_+ < +\infty$. Since $da/dt > 0, d\rho/dt < 0$, we have $a \rightarrow a_+ (\leq +\infty), \rho \rightarrow \rho_+ (\geq 0)$ as $t \rightarrow t_+ - 0$. Suppose $\rho_+ > 0$. Then (13) implies that $a_+ < +\infty$. Then $\dot{a}(t)$ tends to a finite positive limit, and the solution could be continued to the left across $t = t_+$, a contradiction. Therefore it should be the case that $\rho_+ = 0$, and (13) implies that $a_+ = +\infty$. Then, as $t \rightarrow t_+$,

$$\left(\frac{8\pi G}{3}\rho + \frac{c^2\Lambda}{3}\right)a^2 - c^2K \sim \frac{c^2\Lambda}{3}a^2$$

and $\frac{da}{dt} \leq Ca$ with a sufficiently large constant C . Then it is impossible that a blows up with finite t_+ . Therefore we know $t_+ = +\infty$.

We see $a(t) \geq a_0 + \sqrt{\delta}t \rightarrow +\infty$ as $t \rightarrow +\infty$. Then (13) implies $\rho(t) \rightarrow 0$. This completes the proof. \square

In order to observe the asymptotic behaviors of $a(t), \rho(t)$, we put the additional assumption:

(A2) There is a constant γ such that $1 < \gamma$ and $P = O(\rho^\gamma)$ as $\rho \rightarrow 0$.

Under this assumption we can fix the variable ρ^\flat by

$$\rho^\flat = \rho \exp \left[- \int_0^\rho \frac{P/c^2\rho}{1 + P/c^2\rho} \frac{d\rho}{\rho} \right], \quad (25)$$

and then we have

$$\rho^\flat = \rho(1 + O(\rho^{\gamma-1}))$$

as $\rho \rightarrow 0$. Suppose the conditions of Theorem 2 hold. Then, since

$$\begin{aligned} \rho &= \rho_1 a^{-3} (1 + O(a^{-3(\gamma-1)})) \\ &= O(a^{-3}) \end{aligned}$$

as $a \rightarrow \infty$ with a positive constant ρ_1 , we have

$$\left(\frac{8\pi G}{3}\rho + \frac{c^2\Lambda}{3}\right)a^2 - c^2K = \frac{c^2\Lambda}{3}a^2(1 + O(a^{-2}))$$

so that

$$\frac{da}{dt} = c\sqrt{\frac{\Lambda}{3}}a(1 + O(a^{-2})).$$

Hence

$$a = a_1 e^{c\sqrt{\Lambda/3}t} (1 + O(e^{-2c\sqrt{\Lambda/3}t})), \quad (26)$$

$$\rho = \rho_1 a_1^{-3} e^{-3c\sqrt{\Lambda/3}t} (1 + O(e^{-\nu c\sqrt{\Lambda/3}t})) \quad (27)$$

as $t \rightarrow +\infty$ with a suitable positive constant a_1 . Here $\nu = \min(2, 3(\gamma - 1))$.

Remark 4. It follows from Theorem 1 and Theorem 2 that the Einstein's universe (1917) :

$$a(t) = 1, \quad \dot{a}(t) = 0, \quad \rho(t) = \bar{\rho},$$

with

$$4\pi G(\bar{\rho} + 3\bar{P}/c^2) = c^2 \Lambda$$

is unstable. Actually the initial condition

$$a_0 = 1, \quad \dot{a}_0 > 0, \quad \rho_0 = \bar{\rho}$$

satisfies the conditions of Theorem 2, even if \dot{a}_0 is arbitrarily small. Then the solution is infinitely expanding universe when continued to the future. On the other hand, the initial condition

$$a_0 = 1, \quad \dot{a}_0 < 0, \quad \rho_0 = \bar{\rho}$$

satisfies the conditions of Theorem 1, by converting the direction of the time t , even if $|\dot{a}_0|$ is arbitrarily small. Then the solution is a 'Big Crunch' when continued to the future, that is, $t_+ < +\infty$ and $a(t) \rightarrow 0, \rho(t) \rightarrow +\infty$ as $t \rightarrow t_+ - 0$.

If $K \leq 0$, then the condition (24) of Theorem 2 can be dropped, that is, we have

Corollary 2 *Suppose (16): $\dot{a}_0 > 0$. If $K \leq 0$, that is, if (19) holds, then the conclusion of Theorem 2 holds even if (24) is not satisfied*

In fact, suppose $K \leq 0$. Then we have $\dot{a}(t) > 0, d\rho/dt < 0$ a priori for $0 \leq \forall t < t_+$. Thus $\rho(t) \rightarrow \rho_+ (\geq 0), a(t) \rightarrow a_+ (\leq +\infty)$ as $t \rightarrow t_+$. Suppose $\rho_+ > 0$. Then $a_+ < +\infty$ thanks to (13). If $t_+ < +\infty$, then the solution could be continued to the right beyond t_+ , a contradiction. Hence $t_+ = +\infty$. Since $\rho_+ > 0$ is supposed, we should have $a_+ < +\infty$. But (14) and $K \leq 0$ implies

$$\frac{da}{dt} \geq c\sqrt{\frac{\Lambda}{3}}a,$$

which implies $a \geq \text{Const.} e^{c\sqrt{\Lambda/3}t} \rightarrow +\infty$ as $t \rightarrow +\infty$, a contradiction. Therefore we can claim that $\rho_+ = 0$ and there exists $t_0 \in]0, t_+[$ such that

$$4\pi G(\rho(t_0) + 3P(t_0)/c^2) \leq c^2 \Lambda,$$

and the conclusion of Theorem 2 holds. \square

Let us consider the case in which (24) does not hold, that is,

$$4\pi G(\rho_0 + 3P_0/c^2) > c^2\Lambda.$$

Then we have

Corollary 3 *Suppose (A0) and*

$$4\pi G(\rho_0 + 3P_0/c^2) > c^2\Lambda.$$

Then there exists $\epsilon > 0$ depending upon a_0, ρ_0 such that, if $0 < \dot{a}_0 \leq \epsilon$, then $t_+ < +\infty$ and $a(t) \rightarrow 0, \rho(t) \rightarrow +\infty$ as $t \rightarrow t_+ - 0$.

Proof. Note that

$$\frac{d}{d\rho}(\rho + 3P/c^2) = 1 + \frac{3}{c^2} \frac{dP}{d\rho} > 0.$$

So we can take $\rho_* > 0$ such that $\rho_* < \rho_0$ and

$$\delta := 4\pi G(\rho_* + 3P_*/c^2) - c^2\Lambda > 0 \quad \text{with} \quad P_* = P|_{\rho=\rho_*}.$$

Consider the solution $(a^0(t), \dot{a}^0(t), \rho^0(t))$ with the initial data $(a_0, 0, \rho_0)$ and take $T > 0$ sufficiently small so that the solution exists on $[0, T]$ and satisfies $a^0(t) < a_0\Theta^{-1/3}$ for $0 \leq \forall t \leq T$, where Θ is a positive number such that $\rho_*^\flat/\rho_0^\flat < \Theta < 1$. Here we denote $\rho_0^\flat = \rho^\flat|_{\rho=\rho_0}, \rho_*^\flat = \rho^\flat|_{\rho=\rho_*}$. Then, taking a positive ϵ sufficiently small, we can suppose that the solution $(a(t), \dot{a}(t), \rho(t))$ with the initial data (a_0, \dot{a}_0, ρ_0) exists and satisfies $a(t) < a_0\Theta^{-1/3}$ for $0 \leq \forall t \leq T$, provided that $0 < \dot{a}_0 \leq \epsilon$. Moreover we can suppose $\epsilon < \frac{\delta}{3}a_0T$. We claim that there exists $t_0 \in [0, T]$ such that $\dot{a}(t_0) < 0$. Otherwise, $\dot{a}(t) \geq 0$ and $a(t) \geq a_0$ for $0 \leq \forall t \leq T$. Since $a(t) < a_0\Theta^{-1/3}$ for $0 \leq \forall t \leq T$, we have $\rho^\flat(t) > \rho_*^\flat$, therefore $\rho(t) > \rho_*$ for $0 \leq \forall t \leq T$. Hence

$$\frac{d\dot{a}}{dt} = -\frac{1}{3}(4\pi G(\rho + 3P/c^2) - c^2\Lambda)a < -\frac{\delta}{3}a$$

for $0 \leq \forall t \leq T$, and

$$\dot{a}(T) < \dot{a}_0 - \frac{\delta}{3} \int_0^T a dt \leq \dot{a}_0 - \frac{\delta}{3}a_0T \leq \epsilon - \frac{\delta}{3}a_0T < 0,$$

a contradiction to $\dot{a}(T) \geq 0$. Therefore there should exist $t_0 \in [0, T]$ such that $\dot{a}(t_0) < 0$, that is, the universe will become contracting. Moreover since $\rho(t_0) > \rho_*$, we have

$$4\pi G(\rho(t_0) + 3P(t_0)/c^2) > c^2\Lambda.$$

Then Theorem 1 can be applied by converting the direction of the time t . \square

Remark 5. In the case of Corollary 3, there is t_* such that $0 < t_* < t_+$, $\dot{a}(t) > 0$ for $0 \leq \forall t < t_*$ and $\dot{a}(t_*) = 0$. Then, by the uniqueness of the solution of (9a)(9b)(9c) we see that $t_- = 2t_* - t_+$ and $a(t) = a(t_+ + t_- - t)$ for $t_- < t < t_*$, that is, the solution $a(t)$ performs a Big Bang at $t = t_-$ and a Big Crunch at $t = t_+$.

Let us consider the solution continued to the future supposing that $\dot{a}_0 < 0$. By converting the direction of t we can apply Theorem 1 and Corollary 1, that is, either if the condition (18) :

$$4\pi G(\rho_0 + 3P_0/c^2) \geq \Lambda$$

holds or if $K \leq 0$:

$$\left(\frac{8\pi G}{3}\rho_0 + \frac{c^2\Lambda}{3}\right)(a_0)^2 \leq (\dot{a}_0)^2$$

holds, then we have a Big Crunch. When (18) does not hold and $K > 0$, then we can use

Corollary 4 *Suppose (A0) and*

$$4\pi G(\rho_0 + 3P_0/c^2) < c^2\Lambda.$$

Then there exists $\epsilon > 0$ depending upon a_0, ρ_0 such that, if $-\epsilon \leq \dot{a}_0 < 0$, then $t_+ = +\infty$ and $a(t) \rightarrow +\infty, \rho(t) \rightarrow 0$ as $t \rightarrow +\infty$.

Proof. Take $\rho_* > 0$ such that $\rho_0 < \rho_*$ and

$$\delta := c^2\Lambda - 4\pi G(\rho_* + 3P_*/c^2) > 0.$$

It is possible. Consider the solution $(a^o(t), \dot{a}^o(t), \rho^o(t))$ with the initial data $(a_0, 0, \rho_0)$ and take $T > 0$ such that the solution exists and satisfies $a^o(t) > a_0\Theta^{1/3}$ for $0 \leq t \leq T$, where Θ is a positive number such that $\rho_0^b/\rho_*^b < \Theta < 1$. Then, taking $\epsilon > 0$ sufficiently small, we can suppose that the solution $(a(t), \dot{a}(t), \rho(t))$ with the initial data (a_0, \dot{a}_0, ρ_0) exists and satisfies $a(t) > a_0\Theta^{1/3}$ for $0 \leq t \leq T$, provided that $-\epsilon \leq \dot{a}_0 < 0$. We can suppose that $\epsilon < \frac{\delta}{3}a_0\Theta^{1/3}T$. We can claim that $\dot{a}(T) > 0$. In fact, since $a(t) > a_0\Theta^{1/3}$, we have $\rho^b(t) < \rho_*^b$, therefore, $\rho(t) < \rho_*$ for $0 \leq t \leq T$. Hence

$$\frac{d\dot{a}}{dt} = -\frac{1}{3}(4\pi G(\rho + 3P/c^2) - c^2\Lambda)a > \frac{\delta}{3}a$$

and

$$\dot{a}(T) > \dot{a}_0 + \frac{\delta}{3} \int_0^T a dt \geq \dot{a}_0 + \frac{\delta}{3} a_0 \Theta^{1/3} T > 0.$$

Now, since $\dot{a}(T) > 0$ and

$$c^2\Lambda - 4\pi G(\rho(T) + 3P(T)/c^2) > 0,$$

Theorem 2 can be applied to claim that $t_+ = +\infty$ and $a(t) \rightarrow +\infty, \rho(t) \rightarrow 0$ as $t \rightarrow +\infty$. \square

Appendix

Let us observe possible scenarios assuming that the equation of state is $P = 0$.

Integrating (9c) gives

$$\rho = \rho_1 a^{-3}$$

with a positive constant $\rho_1 = \rho_0(a_0)^3$. The Friedman equation (14) turns out to be

$$\left(\frac{da}{dt}\right)^2 = \frac{8\pi G\rho_1}{3} \frac{1}{a} + \frac{c^2\Lambda}{3} a^2 - c^2 K.$$

We shall consider the case $K > 0$, and, suppose

$$0 < \dot{a}_0 < \sqrt{\frac{8\pi G\rho_1}{3} \frac{1}{a_0} + \frac{c^2\Lambda}{3} (a_0)^2}.$$

Let $]t_-, t_+[$ be the maximal interval of existence of the solution in the domain $\{0 < a, |\dot{a}| < \infty, 0 < \rho\}$, and let $]t_-^+, t_+^+[$ be that in the domain $\{0 < a, 0 < \dot{a}, 0 < \rho\}$. When $t \in]t_-^+, t_+^+[$, we can take

$$\frac{da}{dt} = \sqrt{\frac{8\pi G\rho_1}{3} \frac{1}{a} + \frac{c^2\Lambda}{3} a^2 - c^2 K},$$

which can be integrated as

$$I := \int \sqrt{\frac{a}{a^3 - \frac{3K}{\Lambda}a + \frac{8\pi G\rho_1}{c\Lambda}}} da = c\sqrt{\frac{\Lambda}{3}}(t - t_0),$$

where t_0 is arbitrary. Introducing the variable ξ and the parameter α defined by

$$a = \sqrt{\frac{K}{L}}\xi, \quad \alpha = \frac{4\pi G\rho_1}{c^2} \sqrt{\frac{\Lambda}{K^3}},$$

we write

$$I = \int \sqrt{\frac{\xi}{\xi^3 - 3\xi + 2\alpha}} d\xi.$$

Let us consider the following cases:

[[Case-0]]: $0 < \alpha < 1$,

[[Case-1]]: $\alpha = 1$,

[[Case-2]]: $1 < \alpha < +\infty$.

Let us denote

$$f_\alpha(\xi) := \xi^3 - 3\xi + 2\alpha.$$

First we consider the case [Case-1]: $\alpha = 1$. In this case we have

$$f_1(\xi) = (\xi - 1)^2(\xi + 2),$$

and

$$\sqrt{\frac{\xi}{\xi^3 - 3\xi + 2}} = \frac{1}{|\xi - 1|} \sqrt{\frac{\xi}{\xi + 2}}.$$

Therefore, using the variable x defined by

$$x = \sqrt{\frac{\xi}{\xi + 2}},$$

we see the integral I is given as follows:

$$\text{[Case-1.0]: for } \frac{1}{\sqrt{3}} < x < 1$$

$$I = \frac{1}{\sqrt{3}} \log \frac{\sqrt{3}x - 1}{\sqrt{3}x + 1} + \log \frac{1 + x}{1 - x},$$

and

$$\text{[Case-1.1]: for } 0 < x < \frac{1}{\sqrt{3}}$$

$$I = \frac{1}{\sqrt{3}} \log \frac{1 + \sqrt{3}x}{1 - \sqrt{3}x} + \log \frac{1 - x}{1 + x}.$$

The case [Case-1.0] corresponds to $1 < \xi$, and the solution $a(t)$ exists and monotone increases on $] - \infty, +\infty[$. Asymptotic behavior is:

$$\begin{aligned} a(t) &\sim C e^{c\sqrt{\Lambda/3}t} \quad \text{as } t \rightarrow +\infty, \\ a(t) - \bar{a} &\sim C e^{c\sqrt{\Lambda}t} \quad \text{as } t \rightarrow -\infty \end{aligned}$$

with

$$\bar{a} = \left(\frac{4\pi G \rho_1}{c^2 \Lambda} \right)^{1/3}.$$

Here and hereafter C stands for various positive constants.

Note that $(a, \dot{a}, \rho) = (\bar{a}, 0, \rho_0)$, where $\rho_0 = \frac{c^2 \Lambda}{4\pi G}$, is the Einstein's static universe (1917).

On the other hand, [Case-1.1] corresponds to $0 < \xi < 1$, and the solution $a(t)$ exists and monotone increases on $]t_-, +\infty[$, where t_- is finite. Asymptotic behavior is:

$$\begin{aligned} \bar{a} - a(t) &\sim C e^{-c\sqrt{\Lambda}t} \quad \text{as } t \rightarrow +\infty, \\ a(t) &\sim (6\pi G \rho_1)^{1/3} (t - t_-)^{2/3} \quad \text{as } t \rightarrow t_- + 0. \end{aligned}$$

Next we consider the case $\llbracket \text{Case-0} \rrbracket$: $0 < \alpha < 1$. In this case the polynomial f_α has three roots, say, $\xi_1, \xi_2, -(\xi_1 + \xi_2)$, where $0 < \xi_1 < 1 < \xi_2$, so that

$$f_\alpha(\xi) = (\xi - \xi_1)(\xi - \xi_2)(\xi + \xi_1 + \xi_2).$$

Therefore $\xi > 0$ such that $f_\alpha(\xi) > 0$ are divided into two separated intervals:

$\llbracket \text{Case-0.0} \rrbracket$: $0 < \xi < \xi_1$,

$\llbracket \text{Case-0.1} \rrbracket$: $\xi_2 < \xi$.

The case $\llbracket \text{Case-0.0} \rrbracket$ gives a solution $a(t)$ which exists and monotone increases on $]t_-, t_+^+[$, where t_-, t_+^+ are finite. Put $t_* = t_+^+$. Asymptotic behavior is:

$$a(t) = \sqrt{\frac{K}{\Lambda}} \left(\xi_1 - \frac{4c^2\Lambda(1-\xi_1^2)}{\xi_1} (t_* - t)^2 + [(t_* - t)^2]_2 \right) \quad \text{as } t \rightarrow t_* - 0,$$

$$a(t) \sim (6\pi G\rho_1)^{1/3} (t - t_-)^{2/3} \quad \text{as } t \rightarrow t_- + 0.$$

So, we see that the solution $a(t)$ can be continued to the right across $t = t_*$ as a monotone decreasing function as $a(t) = a(2t_* - t)$ for $t > t_*$. Hence $t_+ = 2t_* - t_-$ and

$$a(t) \sim (6\pi G\rho_1)^{1/3} (t_+ - t)^{2/3} \quad \text{as } t \rightarrow t_+ - 0.$$

In the same way we see that the case $\llbracket \text{Case-0.1} \rrbracket$ gives a solution $a(t)$ exists on $] - \infty, +\infty[$, monotone decreases on $] - \infty, t_*[$ and monotone increases on $]t_*, +\infty[$, where $t_* = t_-^+$ is finite. Asymptotic behavior is

$$a(t) \sim C e^{c\sqrt{\Lambda/3}t} \quad \text{as } t \rightarrow +\infty,$$

$$a(t) \sim C' e^{-c\sqrt{\Lambda/3}t} \quad \text{as } t \rightarrow -\infty,$$

while

$$a(t) = \sqrt{\frac{K}{\Lambda}} \left(\xi_2 + \frac{4c^2\Lambda(\xi_2^2 - 1)}{\xi_2} (t - t_*)^2 + [(t - t_*)^2]_2 \right)$$

as $t \rightarrow t_*$, and $a(t) = a(2t_* - t)$.

Finally we see for the case $\llbracket \text{Case-2} \rrbracket$: $1 < \alpha$, the solution $a(t)$ exists and monotone increases on $]t_-, +\infty[$, where t_- is finite. Asymptotic behavior is:

$$a(t) \sim C e^{c\sqrt{\Lambda/3}t} \quad \text{as } t \rightarrow +\infty,$$

$$a(t) \sim (6\pi G\rho_1)^{1/3} (t - t_-)^{2/3} \quad \text{as } t \rightarrow t_- + 0.$$

The scenario of this case is nothing but the so-called ‘Lemaître model’ of [3, pp.70-71, §5.1, (ii)], [1, p.121, §10.8, Case 3(i)] ([5], [6]). In fact there exists a unique $t = t_m$ such that $a(t) = \bar{a}$. Then $d\dot{a}/dt = 0$ at $t = t_m$ and $d\dot{a}/dt < 0$, (> 0) for $t_- < t < t_m$, ($[t_m < t < +\infty]$) respectively. Note that

$$\frac{d\dot{a}}{dt} \sim -C(t - t_-)^{-4/3} \quad \text{as } t \rightarrow t_- + 0$$

and

$$\frac{d\dot{a}}{dt} \sim C' e^{c\sqrt{\Lambda/3}t} \quad \text{as } t \rightarrow +\infty.$$

Since

$$\dot{a} \sim C(t - t_-)^{-1/3} \quad \text{as } t \rightarrow t_- + 0$$

and

$$\dot{a} \sim C' e^{c\sqrt{\Lambda/3}t} \quad \text{as } t \rightarrow +\infty,$$

we see that $\dot{a}(t)$ attains its positive minimum at $t = t_m$. The time period around $t = t_m$ is called ‘coasting period’ in the cosmological context.

Summing up, all possible scenarios with positive K are:

$$\begin{array}{l} \text{AS} \nearrow \text{EE}, \text{EC} \searrow \text{AS}, \text{BB} \nearrow \text{AS}, \text{AS} \searrow \text{BC}, \text{EC} \searrow \nearrow \text{EE}, \text{BB} \nearrow \searrow \text{BC}, \\ \text{BB} \nearrow \text{EE}, \text{EC} \searrow \text{BC} \end{array}$$

except for the Einstein’s static universe (1917). Here AS means ‘asymptotically steady state’, EC ‘exponentially contracting’, BB ‘Big bang’, EE ‘exponentially expanding’, and BC means ‘Big Crunch’.

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